How \( e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \) shows \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \)

To demonstrate the connection between these definitions we will use the binomial

expansion of \( (1 + \frac{1}{n})^n \). The Binomial theorem states \( (a + b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k \). \( \binom{n}{k} \) is a representation of a combination. You may have seen these before in the form \( \binom{n}{k} \). Either way \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \). An in-depth breakdown and explanation of the binomial theorem is provided by Regents Prep here, and an explanation of combinations by Math Forum. A symbol that should be recognized is \( ! \), for factorial. A factorial follows an integer and represents the product of the constant and every natural number less than itself. For example \( 5! = 5(4)(3)(2)(1) = 120 \). As a result a fact that will be utilized is that \( \frac{n!}{(n-1)!} = n \).

Let’s begin by using the binomial theorem to expand \( (1 + \frac{1}{n})^n \). By doing this we arrive at

\[
e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \to \infty} \left[ (\frac{n!}{0!})1^0\left(\frac{1}{n}\right)^0 + (\frac{n!}{1!})1^{n-1}\left(\frac{1}{n}\right)^1 + (\frac{n!}{2!})1^{n-2}\left(\frac{1}{n}\right)^2 + \ldots + (\frac{n!}{n!})1^n\left(\frac{1}{n}\right)^n \right].
\]

By rewriting the right side by using the formula for combinations, raising the \( \frac{1}{n} \) to their respective powers, and taking into account that 1 raised to any power is 1 you arrive at;

\[
\lim_{n \to \infty} \left[ 1 + \left(\frac{n!}{(n-1)!}\right)\left(\frac{1}{n}\right) + \left(\frac{n!}{(n-2)!}\right)\left(\frac{1}{n^2}\right) + \ldots + \left(\frac{1}{n^n}\right) \right].
\]

Using what we know about factorials we can further break down the right side to arrive at;

\[
\lim_{n \to \infty} \left[ 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \ldots + \frac{1}{n^n} \right].
\]

By taking the products of each term on the right side the equality becomes;

\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \to \infty} \left[ 1 + 1 + \frac{(n-1)(n-2)}{2(n^2)} + \ldots + \frac{1}{n^n} \right].
\]

At this point we will be taking the limit of the right side as \( n \) approaches infinity. If you are not familiar with limits or simply need a refresher a basic limit explanation is provided in this Youtube video by FarFromStandard. The main fact to keep in mind is that if \( \lim f(x) = A \) and \( \lim g(x) = B \) then \( \lim f(x) + g(x) = A + B \). With this in mind we can treat each term in the infinite sum as its own
function on the right side so that; \[ \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \left(1 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + 0\right) \]. Recall that \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \). Now imagine letting \( x = 1 \), in this case \( e^1 = 1 + 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \ldots = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots = e \). Through this you can see that \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) = e^x \). So now you can see how \( e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \) shows \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \).